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Multivariate Gabor frames and sampling of entire functions of several variables

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ABSTRACT

We investigate Gabor frames with Gaussian windows in higher dimensions. This problem is equivalent to a sampling problem in Bargmann–Fock space. In contrast to dimension $d = 1$, the frame property is no longer characterized by the density of the lattice. We give sufficient conditions for complex lattices to generate a multivariate Gabor frame with a Gaussian window.

1. Introduction

The investigation of the spanning properties of sets of coherent states goes back to the foundations of quantum mechanics by J. von Neumann [36] and signal analysis by D. Gabor [19].

Precisely, let $\varphi(t) = e^{-\pi t\cdot t}$, $t \in \mathbb{R}^d$ be the Gaussian function of d variables and $\pi(z)\varphi = e^{2\pi i z_2\cdot t}\varphi(t - z_1)$ be a phase-space shift by $z = (z_1, z_2) \in \mathbb{R}^{2d}$. Let $\Lambda \subseteq \mathbb{R}^{2d}$ be a countable set in phase-space $\mathbb{R}^{2d}$ and $\mathcal{G}(\varphi, \Lambda) = \{\pi(\lambda)\varphi\colon\ \lambda \in \Lambda\}$ the corresponding collection of phase-space shifts. The fundamental problem is to determine the spanning properties of $\mathcal{G}(\varphi, \Lambda)$ as a function of Λ. When does $\mathcal{G}(\varphi, \Lambda)$ span a dense subspace of $L^2(\mathbb{R}^d)$ (or some other function space)? When is $\mathcal{G}(\varphi, \Lambda)$ a frame for $L^2(\mathbb{R}^d)$? When is $\mathcal{G}(\varphi, \Lambda)$ linearly independent?

This problem has motivated an impressive body of work in mathematical physics, signal processing, in harmonic and complex analysis, its investigation in applied mathematics is nowadays referred to as Gabor analysis.

Gabor analysis in dimension $d = 1$ has been thoroughly studied from 1970 to 1990, and the fundamental questions have been solved with a fascinating mixture of harmonic analysis and complex analysis. If $\Lambda = \mathbb{Z}^2$, then $\mathcal{G}(\varphi, \mathbb{Z}^2)$ spans $L^2(\mathbb{R})$ by the work of Bargmann et al. [3], but the corresponding expansions (*Gabor expansions*)

$$f = \sum_{k,l\in\mathbb{Z}} c_{kl} e^{2\pi i l t} e^{-\pi(t-k)^2}$$

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are not stable and converge only in a distributional sense [1,27]. For stable expansions one has to use the notion of frames, which formalize non-orthogonal overcomplete expansions. Specifically, $\mathcal{G}(\varphi,\Lambda)$ is called a Gabor frame (sometimes also called a Weyl–Heisenberg frame), if there exist $A, B > 0$, such that for all $f \in L^2(\mathbb{R}^d)$

$$A\|f\|_2^2 \leqslant \sum_{\lambda\in\Lambda} \left|\langle f, \pi(\lambda)\varphi\rangle\right|^2 \leqslant B\|f\|_2^2. \tag{1}$$

If (1) holds, then every $f \in L^2(\mathbb{R}^d)$ possesses the expansion $f = \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)\varphi$ for some coefficient sequence $\mathbf{c} \in \ell^2(\Lambda)$ satisfying $\|\mathbf{c}\|_2 \leqslant C\|f\|_2$. In dimension $d = 1$ the groundbreaking characterization of Gabor frames is due independently to Lyubarskiĭ [34] and Seip [42]: they proved that, *in dimension* $d = 1$, $\mathcal{G}(\varphi,\Lambda)$ is a frame if and only if the lower Beurling density $D^-(\Lambda) > 1$. This result solved a conjecture of Daubechies and Grossmann [11] and covered in particular the case when Λ is a lattice. For many more results and variations we refer to the existing monographs on Gabor analysis and time-frequency analysis [16,17,21].

By contrast, in higher dimensions next to nothing is known about the question which lattices $\Lambda \subseteq \mathbb{R}^{2d}$ generate a Gabor frame $\mathcal{G}(\varphi,\Lambda)$. Gabor frames in higher dimensions seem to be on a different level of difficulty and are completely uncharted territory.

In this article we initiate the study of multivariate Gabor frames and the corresponding sampling and interpolation problems in several complex variables. As in [24] we will restrict the set Λ to be a lattice. We introduce the notion of complex lattices (see [10]) and then derive sufficient conditions for such lattices to generate a Gabor frame $\mathcal{G}(\varphi,\Lambda)$. Complex lattices are amenable to complex variable methods. This restriction allows us to extend the techniques used in the univariate case to treat Gabor frames in higher dimensions. In this way we are able to prove a first, reasonably general result about multivariate Gabor frames with a Gaussian window.

A second goal is to highlight the problem of Gabor frames for complex analysts. The frame problem is only one of many problems of time-frequency analysis that lead to interesting questions in complex analysis. Its solution will certainly require input from complex analysis. For other examples of the interaction between time-frequency analysis and complex analysis see [24,35]

The difference between the univariate case and the multivariate case becomes apparent when we look at the proof strategies for the results of Bargmann and Lyubarskiĭ–Seip. The frame inequality (1) is equivalent to a sampling inequality in the Bargmann–Fock space $\mathcal{F}$ consisting of entire functions with norm $\|F\|_{\mathcal{F}}^2 = \int_{\mathbb{C}^d} |F(z)|^2 e^{-\pi|z|^2}\,dz$. The Bargmann transform maps the phase-space shifts $\pi(\lambda)\varphi$ to the reproducing kernel in $\mathcal{F}$, and the frame inequality (1) is then equivalent to a sampling inequality of the form

$$A\|F\|_{\mathcal{F}}^2 \leqslant \sum_{\lambda\in\Lambda} \left|F(\lambda)\right|^2 e^{-\pi|\lambda|^2} \leqslant B\|F\|_{\mathcal{F}}^2 \quad \text{for all } F \in \mathcal{F}. \tag{2}$$

The difference between dimension $d = 1$ and $d > 1$ is now linked to the difference between complex analysis of one variable and several variables. For entire functions of one complex variable the sampling problem (2) and a related interpolation problem can be approached with the full arsenal of complex variable techniques as exemplified by the books of Boas [7] and Levin [33].

By contrast, sampling and interpolation of entire functions of several complex variables are much harder to come by. The zero set of an analytic function of several complex variables is never discrete. Research has mainly focused on the construction of entire functions whose zero set is a given analytic variety and on growth estimates for such functions, see for instance the books of Ronkin [41] and Lelong and Gruman [31]. Since a lattice in $\mathbb{C}^d$ may be interpreted as an intersection of (countably many) hyperplanes, one may construct an entire function with the generating hyperplanes as the zero set. In fact, we will construct an analog of the classical Weierstrass sigma function that vanishes on a given lattice Λ (thus leading to conditions when (2) is violated). Zero sets consisting of hyperplanes have been studied in [37,39], but we have been unable to make use of any of these results. Another relevant direction is the characterization of discrete sets of interpolation for spaces of entire functions of several variables [6,38,39], but again these results are not yet precise enough for us.

The paper is organized as follows: in Section 2 we summarize the prerequisites about Gabor frames and state some important results of the duality theory. In Section 3 we make the transition to complex variables and rephrase the duality theory in terms of sampling and interpolation in Bargmann–Fock space. In Section 4 we introduce complex lattices and discuss the analogues of the classical Weierstrass sigma-function for complex lattices in several dimensions. In Section 5 we discuss which complex lattices generate Gabor frames by solving a corresponding interpolation problem in the Bargmann–Fock space of several complex variables. We study several examples in dimension $d = 2$ and conclude with a discussion of open problems and indicate further directions.

2. Gabor frames

Given a point $z = (x,\xi) \in \mathbb{R}^{2d}$ in phase space, the time-frequency shift of a function f is defined as $\pi(z)f(t) = e^{2\pi i\xi\cdot t} f(t-x)$, $t \in \mathbb{R}^d$. The mapping $f \in L^2(\mathbb{R}^d) \to V_g f(z) = \langle f, \pi(z)g\rangle$ is the short-time Fourier transform and is one of the central objects of time-frequency analysis. For an introduction see [18,21].

A lattice Λ in $\mathbb{R}^{2d}$ is a discrete, cocompact subgroup; every lattice Λ can be written as $\Lambda = A\mathbb{Z}^{2d}$ for some non-singular real $2d \times 2d$-matrix $A \in \mathrm{GL}(2d, \mathbb{R})$. The size of Λ is the volume of a fundamental domain $s(\Lambda) = |\det A|$. The quantity $s(\Lambda)^{-1}$ counts the average number of lattice points per unit cube and coincides with the usual notions of density.

The *adjoint lattice* is defined by the commutant property as

$$\Lambda^\circ = \{\mu \in \mathbb{R}^{2d}\colon \pi(\lambda)\pi(\mu) = \pi(\mu)\pi(\lambda) \text{ for all } \lambda \in \Lambda\}. \tag{3}$$

If $\Lambda = A\mathbb{Z}^{2d} \subset \mathbb{R}^{2d}$, then the adjoint lattice is given explicitly by

$$\Lambda^\circ = \mathcal{J}\big(A^T\big)^{-1}\mathbb{Z}^{2d}, \tag{4}$$

where A^T is the transpose of A and $\mathcal{J} = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ (consisting of $d \times d$ blocks) is the matrix defining the standard symplectic form [15]. Clearly, (4) implies that

$$s\big(\Lambda^\circ\big) = s(\Lambda)^{-1}. \tag{5}$$

For a fixed $g \in L^2(\mathbb{R}^d)$, the set of phase space shifts $\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g\colon \lambda \in \Lambda\}$ is called a frame for $L^2(\mathbb{R}^d)$ (a *Gabor frame*), if there exist $A, B > 0$, such that

$$A\|f\|_2^2 \leqslant \sum_{\lambda\in\Lambda} \big|\langle f, \pi(\lambda)g\rangle\big|^2 \leqslant B\|f\|_2^2 \quad \text{for all } f \in L^2\big(\mathbb{R}^d\big). \tag{6}$$

If only the right-hand inequality holds, then $\mathcal{G}(g, \Lambda)$ is called a Bessel sequence. We refer to [21] and [8,18] for a detailed discussion of Gabor frames.

The fundamental question of Gabor analysis is the following: *Given a window $g \in L^2(\mathbb{R}^d)$, characterize all lattices $\Lambda \subseteq \mathbb{R}^{2d}$, such that $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$.*

This question is solved for exactly three functions so far, and only in dimension $d = 1$, namely the Gaussian and two related functions [30,34,42]. At this time, nothing is known in higher dimensions.

2.1. Some structure theory

There exists a rich theory for the structure of Gabor frames, and many criteria are known to check when a Gabor system is a frame. We will make use of the following results.

Proposition 1 *(Wexler–Raz biorthogonality relations). Assume that $g \in L^2(\mathbb{R}^d)$ and that $\mathcal{G}(g, \Lambda)$ is a Bessel sequence. Then $\mathcal{G}(g, \Lambda)$ is a frame in $L^2(\mathbb{R}^d)$, if and only if there exists $\gamma \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(\gamma, \Lambda)$ is a Bessel sequence and γ satisfies the biorthogonality relations*

$$\frac{1}{s(\Lambda)}\big\langle \gamma, \pi(\mu)g\big\rangle = \delta_{\mu,0}, \quad \text{for } \mu \in \Lambda^\circ. \tag{7}$$

Proposition 2 *(Density theorem).*

(a) *If $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$, then $s(\Lambda) \leqslant 1$.*
(b) *If $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ and if in addition $g \in \mathcal{S}(\mathbb{R}^d)$, then even $s(\Lambda) < 1$.*

Propositions 1 and 2 are well-known. Elementary proofs involve only the Poisson summation formula [28,29], abstract proofs require operator theory [12]. See [26] and [5] for historical accounts of both theorems and extended lists of references and [21] for the general theory.

Proposition 2(b) is known as the Balian–Low theorem. Although it is stated in the literature only for symplectic lattices [23], it holds for arbitrary lattices as an immediate consequence of the following stability result for Gabor frames: *For $g \in \mathcal{S}(\mathbb{R}^d)$ the set of lattices such that $\mathcal{G}(g, \Lambda)$ is a frame is open* [14]. In other words, if $\Lambda = A\mathbb{Z}^{2d}$, $g \in \mathcal{S}(\mathbb{R}^d)$, and $\mathcal{G}(g, \Lambda)$ is a frame, then there exists a neighborhood V of A in $\mathrm{GL}(2d, \mathbb{R})$, such that $\mathcal{G}(g, A'\mathbb{Z}^{2d})$ is a frame for every matrix $A' \in V$. If $\mathcal{G}(g, A\mathbb{Z}^{2d})$ were a frame and $|\det A| = 1$, then we could choose a matrix A' in a small neighborhood of A with $|\det A'| > 1$ such that $\mathcal{G}(g, A'\mathbb{Z}^{2d})$ is a frame. But this would contradict the density theorem (Proposition 2(a)).

Another useful characterization of Gabor frames does not even require any inequalities [22]. For its formulation we denote by $M^\infty(\mathbb{R}^d)$ the subspace of tempered distributions with bounded short-time Fourier transform. Precisely, fix the Gaussian $\varphi(t) = e^{-\pi t\cdot t}$, $t \in \mathbb{R}^d$, then $M^\infty(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d)\colon V_\varphi f \in L^\infty(\mathbb{R}^{2d})\}$ with norm $\|f\|_{M^\infty} = \|V_\varphi f\|_\infty$.

Proposition 3. *(See [22].) Let $g \in \mathcal{S}(\mathbb{R}^d)$ (or more generally $V_\varphi g \in L^1(\mathbb{R}^{2d})$). Then $\mathcal{G}(g, \Lambda)$ is a frame, if and only if the mapping $f \in M^\infty(\mathbb{R}^d) \to V_g f|_\Lambda \in \ell^\infty(\Lambda)$ (the so-called coefficient operator) is one-to-one on $M^\infty(\mathbb{R}^d)$, i.e., if for $f \in M^\infty(\mathbb{R}^d)$ we have $\langle f, \pi(\lambda)g\rangle = 0$ for all $\lambda \in \Lambda$, then $f = 0$.*

3. Gaussians and complex variables

In this section we explain the transition from Gabor frames to sampling and interpolation in the Bargmann–Fock space. We study only Gabor frames with the Gaussian window $\varphi(t) = e^{-\pi t\cdot t}$, $t \in \mathbb{R}^d$.

For $d = 1$ Lyubarskiĭ [34] and Seip [42] proved that $\mathcal{G}(\varphi, \Lambda)$ is a frame, if and only if $s(\Lambda) < 1$. Thus in dimension $d = 1$ the size (or density) of the lattice is the only parameter that determines the frame property.

In higher dimensions $d > 1$, however, the density alone cannot characterize the frame property. For example, for $d = 2$ choose the lattice $\Lambda = \mathbb{Z}^2 \times (\mathbb{Z} \times \epsilon\mathbb{Z})$. Then clearly $s(\Lambda) = \epsilon$ can be chosen arbitrarily small. The Gabor system $\mathcal{G}(\varphi, \Lambda) \subseteq L^2(\mathbb{R}^2)$ is the tensor product of the Gabor systems $\mathcal{G}(\varphi, \mathbb{Z} \times \epsilon\mathbb{Z})$ and $\mathcal{G}(\varphi, \mathbb{Z}^2)$ in $L^2(\mathbb{R})$. Since $\mathcal{G}(\varphi, \mathbb{Z}^2)$ is not a frame by Proposition 2(b), the tensor product cannot be a frame either.

The failure to characterize multivariate sets of sampling by their density is not surprising, a similar phenomenon occurs in the sampling theory of entire functions of exponential type [43].

To make the transition to complex analysis, we recall that the Bargmann–Fock space $\mathcal{F}$ consists of all entire functions F of d complex variables $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ such that

$$\|F\|_{\mathcal{F}}^2 = \int_{\mathbb{C}^d} \left|F(z)\right|^2 e^{-\pi|z|^2}\,dz < \infty. \tag{8}$$

The inner product in $\mathcal{F}$ is $\langle F, G\rangle_{\mathcal{F}} = \int_{\mathbb{C}^d} F(z)\overline{G(z)}e^{-\pi|z|^2}dz$, and dz is the Lebesgue measure on $\mathbb{C}^d$. We write $z \cdot w = \sum_{j=1}^d z_j w_j$ for $z, w \in \mathbb{C}^d$ and $|z|^2 = z \cdot \bar{z}$.

Next we consider the Bargmann transform of a function $f \in L^2(\mathbb{R}^d)$, which is defined by

$$\mathcal{B}f(z) = F(z) = 2^{d/4} e^{-\pi z^2/2} \int_{\mathbb{R}^d} f(t) e^{-\pi t\cdot t} e^{2\pi t\cdot z}\,dt, \quad z \in \mathbb{C}^d. \tag{9}$$

The Bargmann transform is a unitary mapping between $L^2(\mathbb{R}^d)$ and $\mathcal{F}$ [2,18]. The Bargmann transform is a short-time Fourier transform in disguise, precisely, for $z = (x, \xi) \in \mathbb{R}^{2d}$ corresponding to $z = x + i\xi \in \mathbb{C}^d$, we have

$$V_\varphi f(\bar{z}) = e^{\pi i x\cdot\xi} \mathcal{B}f(z) e^{-\pi|z|^2/2}. \tag{10}$$

The Fock representation of $\mathbb{C}^d$ on $\mathcal{F}$ is defined as

$$\beta(z)F(w) = e^{i\pi x\cdot\xi} e^{\pi wz} F(w - \bar{z}) e^{-\pi|z|^2/2}. \tag{11}$$

Then every $\beta(z)$ is a unitary operator on $\mathcal{F}$, and the Bargmann transform intertwines $\beta(z)$ and the time-frequency shifts $\pi(z)$:

$$\beta(z)\mathcal{B} = \mathcal{B}\pi(z), \quad z \in \mathbb{C}^d. \tag{12}$$

Since $|\langle f, \pi(\bar{z})\varphi\rangle| = |V_\varphi f(\bar{z})| = |\mathcal{B}f(z)|e^{-\pi|z|^2/2}$, the set $\mathcal{G}(\varphi, \overline{\Lambda})$ is a frame, if and only if

$$\sum_{\lambda\in\Lambda} \left|F(\lambda)\right|^2 e^{-\pi|\lambda|^2} \asymp \|F\|_{\mathcal{F}}^2, \quad \text{for all } F \in \mathcal{F}. \tag{13}$$

In other words, $\mathcal{G}(\varphi, \overline{\Lambda})$ is a frame, if and only if Λ is a set of sampling for $\mathcal{F}$.

Next we translate the general characterizations of Gabor frames by using the Bargmann transform. For the formulation we need the Bargmann–Fock space $\mathcal{F}^\infty$ consisting of all entire functions F on $\mathbb{C}^d$ satisfying $|F(z)| \leqslant Ce^{\pi|z|^2/2}$ for all $z \in \mathbb{C}^d$. It can be shown that $\mathcal{F}^\infty$ is the range of the modulation space M^∞ under the Bargmann transform, i.e., $\mathcal{F}^\infty = \mathcal{B}(M^\infty)$, see for instance [20].

Proposition 4. *Let $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice with adjoint lattice Λ°. Then the following are equivalent*:

(i) *$\mathcal{G}(\varphi, \overline{\Lambda})$ is a frame for $L^2(\mathbb{R}^d)$.*
(ii) *Λ is a set of sampling for $\mathcal{F}$.*
(iii) *There exists an interpolating function $G \in \mathcal{F}$, such that*

$$G(\mu) = \delta_{\mu,0}, \quad \textit{for all } \mu \in \Lambda^\circ. \tag{14}$$

(iv) *Λ is a set of uniqueness for $\mathcal{F}^\infty$, i.e., if $F \in \mathcal{F}^\infty$ and $F(\lambda) = 0$ for all $\lambda \in \Lambda$, then $F \equiv 0$.*

Proof. The equivalence of (i) and (ii) follows by comparing (6) and (13).

By Proposition 1 $\mathcal{G}(\varphi, \overline{\Lambda})$ is a frame, if and only if there exists a $\gamma \in L^2(\mathbb{R}^d)$ such that $\delta_{\mu,0} = |\langle \gamma, \pi(\mu)\varphi\rangle| = |\mathcal{B}\gamma(\bar{\mu})|e^{-\pi|\mu|^2/2}$ for $\mu \in \overline{\Lambda^\circ}$. For the equivalence (i) $\Leftrightarrow$ (iii) just set $G = \mathcal{B}\gamma$.

The equivalence (ii) $\Leftrightarrow$ (iv) follows from Proposition 3 and the surjectivity of $\mathcal{B}$ from M^∞ onto $\mathcal{F}^\infty$. $\square$

The equivalence of conditions (i)–(iii) is not new and is underlying the work of Lyubarskiĭ [34] and Seip [42].

The equivalence of (ii) and (iv) is easy to establish directly for entire functions of one variable with standard methods [7,33]. However, for entire functions of several variables the criterion seems to be new. Our proof of Proposition 3 and Proposition 4 in [22] requires all known results about the structure and duality of Gabor frames. It might be a challenge to give a direct proof using only complex variable arguments.

4. Weierstrass sigma functions and complex lattices

In dimension $d = 1$ a solution of the interpolation problem (14) can be written explicitly by means of the classical Weierstrass sigma functions. In this section we construct analogues of the Weierstrass sigma function for a class of lattices in $\mathbb{C}^d$.

We recall the following fact about sigma functions that is fundamental for the understanding of Gabor frames in $L^2(\mathbb{R})$.

Proposition 5. *Let $L \subseteq \mathbb{C}$ be a lattice in $\mathbb{C}$ of size $s(L)$. Then there exists an entire function $\sigma = \sigma_L$ such that*

- *$\sigma_L(\kappa) = 0$ for $\kappa \in L$, and*
- *$|\sigma_L(z)| \leqslant C(L)e^{\frac{\pi}{2s(L)}|z|^2}$ for all $z \in \mathbb{C}$.*

The function σ_L is a modification of the classical Weierstrass sigma-function, and is given explicitly by

$$\sigma_L(z) = \left(z \prod_{\lambda \in \Lambda\setminus\{0\}} \left(1 - \frac{z}{\lambda}\right) e^{\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}}\right) e^{az^2}, \tag{15}$$

for a suitable complex number $a \in \mathbb{C}$. Using the quasi-periodicity of the classical Weierstrass σ-function, one can show that the parameter a can be chosen so that the function $|\sigma_L(z)|e^{-\frac{\pi}{2s(\Lambda)}|z|^2}$ is L-periodic, whence the growth estimate follows. The argument goes back to Hayman [25] and is given in detail in [24].

Note that the interpolation problem (14) is solved by the function $G(z) = \sigma_L(z)/z$ and that G satisfies the same growth estimate as σ_L.

While an arbitrary lattice in $\mathbb{R}^{2d} \simeq \mathbb{C}^d$ is of the form $\Lambda = \mathcal{A}\mathbb{Z}^{2d}$ for some invertible, real-valued $2d \times 2d$-matrix $\mathcal{A} \in \mathrm{GL}(2d, \mathbb{R})$, we will only consider complex lattices. These are adapted to the complex structure.

Essentially a complex lattice is of the form $\Lambda = A(\mathbb{Z}^d + i\mathbb{Z}^d)$ for some invertible complex-valued $d \times d$-matrix A, $A \in \mathrm{GL}(d, \mathbb{C})$, see for instance [10]. For a slightly more general definition let $L \subseteq \mathbb{C}$ be a normalized lattice in $\mathbb{C}$, i.e., L is spanned by two complex numbers ω_1 and ω_2, such that $s(L) = \frac{i}{2}(\omega_1\overline{\omega_2} - \overline{\omega_1}\omega_2) = i\,\mathrm{Im}\,\omega_1\overline{\omega_2} = 1$. Written in real form, $L = B\mathbb{Z}^2$ for a 2×2-matrix $B \in \mathrm{SL}(2, \mathbb{R})$.

Definition 1. A complex lattice is a lattice of the form

$$\Lambda = A\left(\bigoplus_{j=1}^{d} L_j\right)$$

for some $A \in \mathrm{GL}(d, \mathbb{C})$ and normalized lattices $L_j \subseteq \mathbb{C}$, $j = 1, \ldots, d$. The size of a complex lattice is $s(\Lambda) = |\det A|^2$, see [10].

For complex lattices we may write down explicit d-dimensional analogues of the Weierstrass σ-function.

Proposition 6. *Let $\Lambda = A(\bigoplus_{j=1}^d L_j) \subseteq \mathbb{C}^d$ be a complex lattice.*

(a) *Then there exists an entire function G_Λ solving the interpolation problem*

$$G_\Lambda(\lambda) = \delta_{\lambda,0} \quad \text{for } \lambda \in \Lambda \tag{16}$$

satisfying the growth estimate

$$\left|G_\Lambda(z)\right| \leqslant Ce^{\pi\|A^{-1}\|_{op}^2|z|^2/2}.$$

(b) *Similarly, there exists an entire function σ_Λ such that $\sigma_\Lambda(\lambda) = 0$ for all $\lambda \in \Lambda$ with growth $|\sigma_\Lambda(z)| \leqslant Ce^{\pi \|A^{-1}\|_{op}^2 |z|^2/2}$. (As usual $\|A\|_{op}$ denotes the largest singular value of A.)*

Proof. (a) For a lattice L in $\mathbb{C}$ the interpolation problem (16) is solved by the function $s_L(z)/z$, where s_L is the (modified) Weierstrass sigma-function (15) of Proposition 5. For a direct sum of 1-dimensional lattices $\Gamma_0 = \bigoplus_{j=1}^d L_j$ we set

$$\sigma_0(z) = \sigma_0(z_1, \dots, z_d) = \prod_{j=1}^d \frac{\sigma_{L_j}(z_j)}{z_j}.$$

Finally for an arbitrary complex lattice we set

$$G_\Lambda(z) = \sigma_0\bigl(A^{-1}z\bigr).$$

If $\lambda = A(\kappa_1, \dots, \kappa_d) \in \Lambda$ with $\kappa_j \in L_j$, then

$$G_\Lambda(\lambda) = \sigma_0\bigl(A^{-1}\lambda\bigr) = \sigma_0(\kappa_1, \dots, \kappa_d)$$
$$= \prod_{j=1}^d \frac{\sigma_{L_j}(\kappa_j)}{\kappa_j} = \prod_{j=1}^d \delta_{\kappa_j,0} = \delta_{\lambda,0}.$$

As for the growth, we have $|\sigma_{L_j}(z_j)| \leqslant C_j e^{\frac{\pi}{2}|z_j|^2}$ and thus $|\sigma_0(z_1, \dots, z_d)| \leqslant Ce^{\frac{\pi}{2}\sum |z_j|^2} = Ce^{\frac{\pi}{2}|z|^2}$. Consequently,

$$\bigl|G_\Lambda(z)\bigr| = \bigl|\sigma_0\bigl(A^{-1}z\bigr)\bigr| \leqslant Ce^{\frac{\pi}{2}|A^{-1}z|^2} \leqslant Ce^{\frac{\pi}{2}\|A^{-1}\|_{op}^2|z|^2}.$$

(b) is similar by using $\sigma_0(z) = \prod_{j=1}^d \sigma_{L_j}(z_j)$ and $\sigma(z) = \sigma_0(A^{-1}z)$. $\square$

Remark. Note that the zero set of σ_Λ is the union of analytic hyperplanes in $\mathbb{C}^d$; the lattice points are exactly the intersections of such hyperplanes and correspond to the singular points of the zero set. The general construction of Weierstrass sigma functions whose zero set is a given analytic manifold is addressed in the books [31,41].

For a given lattice Λ there are many associated sigma functions; these depend on the choice of A or equivalently on the choice of a basis for Λ. The problem is to find a sigma function of minimal growth. This question is closely related to the famous problem of finding short vectors in a lattice or a basis consisting of short vectors [9,32]. We will pursue this point of view in a subsequent work.

In order to apply Proposition 4, we need to compute the adjoint lattice of a complex lattice Λ.

Lemma 7. *If $\Lambda = A\Gamma_0$ for $A \in \mathrm{GL}(d, \mathbb{C})$ and $\Gamma_0 = \bigoplus_{j=1}^d L_j$, then $\Lambda^\circ = (A^*)^{-1}\Gamma_0$.*

Proof. By definition

$$\Lambda^\circ = \bigl\{\mu \in \mathbb{C}^d\colon\ \beta(\lambda)\beta(\mu) = \beta(\mu)\beta(\lambda) \text{ for all } \lambda \in \Lambda\bigr\}.$$

Then for $F \in \mathcal{F}$ we obtain that

$$\beta(\lambda)\beta(\mu)F(z) - \beta(\mu)\beta(\lambda)F(z) = e^{\pi i(\lambda_1\cdot\lambda_2+\mu_1\cdot\mu_2)}\bigl(e^{-\pi\mu\cdot\bar\lambda} - e^{-\pi\lambda\cdot\bar\mu}\bigr)e^{\pi(\lambda+\mu)\cdot z}F(z - \bar\lambda - \bar\mu)e^{-\pi(|\lambda|^2+|\mu|^2)/2},$$

and so $\beta(\lambda)$ commutes with $\beta(\mu)$ for all $\lambda \in \Lambda$, if and only if

$$e^{\pi(\lambda\cdot\bar\mu-\mu\cdot\bar\lambda)} = e^{2\pi i\,\mathrm{Im}\,\lambda\cdot\bar\mu} = 1 \qquad \text{for all } \lambda \in \Lambda,$$

in other words, $\mathrm{Im}\,\lambda\cdot\bar\mu \in \mathbb{Z}$ for all $\lambda \in \Lambda$.

Assume first that $\Lambda = A(\mathbb{Z}^d + i\mathbb{Z}^d)$, then $\lambda \in \Lambda$ is of the form $\lambda = A(k + il)$ for $k, l \in \mathbb{Z}^d$. We find that $\mu \in \Lambda^\circ$, if and only if $\mathrm{Im}\,A^T\bar\mu\cdot(k + il) \in \mathbb{Z}$ for all $k, l \in \mathbb{Z}^d$, if and only if $\mathrm{Im}\,\overline{A^*\mu}\cdot(k + il) \in \mathbb{Z}$ for all $k, l \in \mathbb{Z}^d$, if and only if $\overline{A^*\mu} \in \mathbb{Z}^d + i\mathbb{Z}^d$, if and only if $\mu \in (A^*)^{-1}(\mathbb{Z}^d + i\mathbb{Z}^d)$.

Next assume $\Lambda = A\Gamma_0$ and $\Gamma_0 = \bigoplus L_j$. Let $\lambda = A(\kappa_1, \dots, \kappa_d) \in A\Gamma_0$ and $\mu = (A^*)^{-1}(\rho_1, \dots, \rho_d) \in (A^*)^{-1}\Gamma_0$ with $\kappa_j, \rho_j \in L_j$. Then $\mathrm{Im}\,\lambda\cdot\bar\mu = \sum_{j=1}^d \mathrm{Im}\,\kappa_j\overline{\rho_j}$. Each lattice $L_j \subseteq \mathbb{C}$ is spanned by complex numbers $\omega_1^{(j)}, \omega_2^{(j)} \in \mathbb{C}$ and $\kappa_j = k_1\omega_1^{(j)} + k_2\omega_2^{(j)}$ and $\rho_j = l_1\omega_1^{(j)} + l_2\omega_2^{(j)}$, $k_j, l_j \in \mathbb{Z}$. Then $\mathrm{Im}\,\kappa_j\overline{\rho_j} \in \mathbb{Z}$ for each j. Consequently $(A^*)^{-1}\Gamma_0 \subseteq \Lambda^\circ$. Since $s(\Lambda^\circ) = |\det A|^{-1} = s((A^*)^{-1}\Gamma_0)$, we must have $\Lambda_0 = (A^*)^{-1}\Gamma_0$. $\square$

5. Gabor frames with complex lattices

In this section we formulate a sufficient condition for a Gabor system $\mathcal{G}(\varphi, \Lambda)$ to form a frame. We first state a simple invariance property of sets of sampling for $\mathcal{F}$.

Lemma 8. *Let $U \in \mathcal{U}(d)$ be a unitary $d \times d$-matrix and $\Lambda' = U\Lambda$. Then Λ is a set of sampling for $\mathcal{F}$, if and only if Λ' is a set of sampling for $\mathcal{F}$.*

Proof. Set $\mathcal{T}_U F(z) = F(Uz)$ for $U \in \mathcal{U}(d)$. Then $\mathcal{T}_U$ is a unitary operator on $\mathcal{F}$, and

$$\begin{aligned}\sum_{\lambda\in\Lambda} \big|F(U\lambda)\big|^2 e^{-\pi|U\lambda|^2} &= \sum_{\lambda\in\Lambda} \big|\mathcal{T}_U F(\lambda)\big|^2 e^{-\pi|\lambda|^2} \\ &\asymp \|\mathcal{T}_U F\|_{\mathcal{F}}^2 = \|F\|_{\mathcal{F}}^2.\end{aligned}$$

If Λ is a set of sampling for $\mathcal{F}$, then so is $\Lambda' = U\Lambda$, and conversely. $\square$

Let $A \in \mathrm{GL}(d, \mathbb{C})$. Using the Gram–Schmidt orthogonalization (also called the Iwasawa decomposition), we may factor A into a unitary matrix and an upper triangular matrix. Precisely, there exists a unitary $d \times d$-matrix $U \in \mathcal{U}(d)$ and an upper triangular matrix S such that

$$A = US.$$

By choosing U appropriately, we may assume that

$$S = \begin{pmatrix} \gamma_1 & * & \dots & * \\ 0 & \gamma_2 & \dots & * \\ \vdots & & & \\ 0 & \dots & 0 & \gamma_d \end{pmatrix} \tag{17}$$

with the diagonal elements $\gamma_j > 0$. This factorization is unique, and we call the numbers γ_j, $j = 1, \dots, d$, the characteristic indices of A. With this factorization in place, we may now formulate our main theorem.

Theorem 9. *Let $\Lambda = A\mathbb{Z}^{2d}$ be a complex lattice in $\mathbb{C}^d$ and let γ_j be the characteristic indices of A defined in* (17).
If $0 < \gamma_j < 1$ for $j = 1, \dots, d$, then Λ is a set of sampling for $\mathcal{F}$, and the set $\mathcal{G}(\varphi, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$.

Proof. By Lemma 8 we may assume without loss of generality that the lattice is of the form $\Lambda = S\Gamma_0$ for some upper triangular matrix $S \in \mathrm{GL}(d, \mathbb{C})$. Consequently the adjoint lattice is $\Lambda^\circ = (S^*)^{-1}\Gamma_0$.

If S is the upper triangular matrix of (17), then

$$\big(S^*\big)^{-1} = \begin{pmatrix} \gamma_1^{-1} & 0 & \dots & 0 \\ * & \gamma_2^{-1} & 0 & \vdots \\ \vdots & & & 0 \\ * & \dots & * & \gamma_d^{-1} \end{pmatrix}.$$

By assumption we have $\gamma_j^{-1} > 1$ for $j = 1, \dots, d$.

Now define

$$G(z) = \prod_{j=1}^{d} \frac{\sigma_{L_j}(\gamma_j z_j)}{z_j}.$$

Then by Proposition 5 this σ-function satisfies the growth estimate

$$\big|G(z)\big| \leqslant C \prod_{j=1}^{d} e^{\pi\gamma_j^2|z_j|^2/2}.$$

Since $\gamma_j < 1$, we find that G is an entire function in $\mathcal{F}$.

Next let $\mu \in \Lambda^\circ$, $\mu = (S^*)^{-1}\kappa$ for $\kappa = (\kappa_1, \dots, \kappa_d)$ with $\kappa_j \in L_j \subseteq \mathbb{C}$. Since S^* is lower triangular, the ℓ-th component of μ is $\mu_\ell = ((S^*)^{-1}\kappa)_\ell = \gamma_\ell^{-1}\kappa_\ell + \sum_{j<\ell} s_{\ell,j}\kappa_j$. Write $\kappa = (0, \dots 0, \kappa_J, \kappa_{J+1}, \dots, \kappa_d)$ where J is chosen so that $\kappa_J \neq 0$, but $\kappa_j = 0$ for $j < J$. Then $\mu_J = \gamma_J^{-1}\kappa_J$, and thus

$$\begin{aligned} G(\mu) &= \prod_{j=1}^{d} \mu_j^{-1} \sigma_{L_j}(\gamma_j \mu_j) \\ &= \frac{\sigma_{L_J}(\kappa_J)}{\mu_J} \prod_{j \neq J} \mu_j^{-1} \sigma_{L_j}(\gamma_j \mu_j) = 0. \end{aligned}$$

Clearly, if $\mu = 0$, then $G(0) \neq 0$. By Proposition 4 the set $\mathcal{G}(\varphi, \Lambda)$ is therefore a Gabor frame. $\square$

Using a similar factorization of $A = VL$ with $V \in \mathcal{U}(d)$ and a lower triangular matrix L, we obtain a similar sufficient condition.

Corollary 10. *Let Λ be a complex lattice with $A \in \mathrm{GL}(d, \mathbb{C})$. Assume that $A = VL$ for some $V \in \mathcal{U}(d)$ and a lower triangular matrix L with positive entries $\gamma_j > 0$ on the diagonal. If $\gamma_j < 1$ for $j = 1, \ldots, d$, then $\mathcal{G}(\varphi, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$.*

Remarks. 1. The set of all phase-space lattices can be identified with the quotient $\mathrm{GL}(2d, \mathbb{R})/\mathrm{SL}(2d, \mathbb{Z})$ and thus has dimension $4d^2$.

The class of complex lattices in $\mathbb{C}^d$ has real dimension $2d^2 + 2d$, because the real dimension of $\mathrm{GL}(d, \mathbb{C})/\mathrm{SL}(d, \mathbb{Z} + i\mathbb{Z})$ is $2d^2$ and $\bigoplus_{j=1}^d L_j$ has an additional $2d$ free parameters.

2. Theorem 9 sheds some light on the existence problem of Gabor frames for general lattices. Bekka [4] showed that for every lattice $\Lambda \subseteq \mathbb{R}^{2d}$ with $s(\Lambda) \leqslant 1$, there exists some $g \in L^2(\mathbb{R}^d)$, such that $\mathcal{G}(g, \Lambda)$ is a Gabor frame. For $s(\Lambda) < 1$ it is expected that there exist arbitrarily smooth functions g, such that $\mathcal{G}(g, \Lambda)$ is a frame. Theorem 9 asserts that for complex lattices with small characteristic values one may even use the Gaussian to obtain a Gabor frame.

5.1. Complex lattices in dimension $d = 2$

At first glance one might suspect that the condition $\gamma_j < 1$ is also necessary for $\mathcal{G}(\varphi, \Lambda)$ to be a frame. However, looking at a few examples in dimension $d = 2$ shows that the situation is much more delicate.

We assume that $A = \left(\begin{smallmatrix} \gamma_1 & b \\ 0 & \gamma_2 \end{smallmatrix}\right)$ for unique $\gamma_j > 0$. Since the matrix A and the matrix $A' = A\left(\begin{smallmatrix} 1 & \kappa \\ 0 & 1 \end{smallmatrix}\right)$ for $\kappa \in L_2$ generate the same lattice $\Lambda = A(L_1 \oplus L_2) = A'(L_1 \oplus L_2)$, there is no loss of generality to assume that b is close to the origin. For example, if $L_1 = L_2 = \mathbb{Z} + i\mathbb{Z}$, then we may take $b = b_1 + ib_2$ with $|b_j| \leqslant \gamma_1/2$.

Proposition 11. *Let $\Lambda = A(L_1 \oplus L_2)$ be a complex lattice in $\mathbb{C}^2$ determined by $A = \left(\begin{smallmatrix} \gamma_1 & b \\ 0 & \gamma_2 \end{smallmatrix}\right)$. Then:*

(i) *If $\gamma_1 < 1$ and $\gamma_2 < 1$, then $\mathcal{G}(\varphi, \Lambda)$ is a frame.*
(ii) *If $\gamma_2 \geqslant 1$, then $\mathcal{G}(\varphi, \Lambda)$ is not a frame.*
(iii) *If $\gamma_1 \geqslant 1$, $\gamma_2 < 1$, and $\gamma_1\gamma_2 < (\gamma_2^2 + |b|^2)^{1/2} < 1$, then $\mathcal{G}(\varphi, \Lambda)$ is a frame.*
(iv) *If $\gamma_1 \geqslant 1$, $\gamma_2 < 1$, $\gamma_1\gamma_2 \geqslant (\gamma_2^2 + |b|^2)^{1/2}$, then $\mathcal{G}(\varphi, \Lambda)$ is not a frame.*

Proof. (i) was proved in Theorem 9.

(ii) Consider the function $\sigma(z_1, z_2) = \sigma_{L_2}(\gamma_2^{-1} z_2)$. Then σ vanishes on Λ and $|\sigma(z)| \leqslant C e^{\pi \gamma_2^{-2} |z_2^2|/2} \leqslant C e^{\pi |z|^2/2}$, because $\gamma_2 \geqslant 1$, and so $\sigma \in \mathcal{F}^\infty$. This means that Λ is not a set of uniqueness for $\mathcal{F}^\infty$, and thus $\mathcal{G}(\varphi, \Lambda)$ cannot be a frame by Proposition 4.

To prove (iii) and (iv), we write A as VL for some lower triangular matrix. Let $V = \left(\begin{smallmatrix} u & v \\ -\bar{v} & \bar{u} \end{smallmatrix}\right)$ with

$$u = \frac{\gamma_2}{(\gamma_2^2 + |b|^2)^{1/2}} \quad \text{and} \quad v = -\frac{b}{(\gamma_2^2 + |b|^2)^{1/2}}.$$

Then V is unitary and

$$V \begin{pmatrix} \gamma_1 & b \\ 0 & \gamma_2 \end{pmatrix} = \begin{pmatrix} \frac{\gamma_1\gamma_2}{(\gamma_2^2 + |b|^2)^{1/2}} & 0 \\ \frac{\bar{b}\gamma_1}{(\gamma_2^2 + |b|^2)^{1/2}} & (\gamma_2^2 + |b|^2)^{1/2} \end{pmatrix}.$$

(iii) The assumptions state that $\frac{\gamma_1\gamma_2}{(\gamma_2^2 + |b|^2)^{1/2}} < 1$ and $(\gamma_2^2 + |b|^2)^{1/2} < 1$, so by Corollary 10, $\mathcal{G}(\varphi, \Lambda)$ is a frame.

(iv) If $\rho_1 = \frac{\gamma_1\gamma_2}{(\gamma_2^2 + |b|^2)^{1/2}} \geqslant 1$, then the function $F(z_1, z_2) = \sigma_{L_1}(\rho_1^{-1} z_1)$ is in $\mathcal{F}^\infty$ and vanishes on the lattice $VA\Gamma_0$. Again by Proposition 4, $\mathcal{G}(\varphi, \Lambda)$ cannot be a frame. $\square$

Items (iii) and (iv) can be proved directly by finding a suitable sigma function. Set $\Delta = (\gamma_2^2 + |b|^2)^{1/2}$. For (iii) set

$$G(z_1, z_2) = \frac{\sigma_{L_1}(\frac{\gamma_1\gamma_2}{\Delta^2}(\gamma_2 z_1 - b z_2))}{\gamma_2 z_1 - b z_2} \frac{\sigma_{L_2}(\bar{b} z_1 + \gamma_2 z_2)}{\bar{b} z_1 + \gamma_2 z_2}.$$

Then one can check that $G(\mu) = \delta_{\mu,0}$ for $\mu \in \Lambda^\circ = (A^*)^{-1}(L_1 \oplus L_2)$ and that $G \in \mathcal{F}$, if and only if, $\gamma_1\gamma_2 < \Delta < 1$.

For (iv) we set $\sigma_\Lambda(z_1, z_2) = \sigma_{L_1}(\gamma_1^{-1}(z_1 - \frac{b}{\gamma_2} z_2))$. This function vanishes on Λ and is in $\mathcal{F}^\infty$, if and only if $\Delta \leqslant \gamma_1\gamma_2$.

At this time the case $\gamma_1\gamma_2 < 1 \leqslant (\gamma_2^2 + |b|^2)^{1/2}$ remains open, although we can already show for several families of lattices whether they generate Gabor frames or not.

5.2. Open problems

1. The analysis of this paper is quite limited, and several difficult questions remain open. Clearly the arguments of Theorem 9 and Proposition 11 can be pushed further. In higher dimensions it will be important to use a suitable basis for Λ consisting of short vectors, so-called reduced basis [9,32]. For complex lattices we have some hope to classify those lattices that generate a Gabor frame.

On the other hand, for real lattices $\mathcal{A}\mathbb{Z}^{2d}$ for general $\mathcal{A} \in \mathrm{GL}(2d, \mathbb{R})$ the field of speculation is wide open. We cannot even guess, let alone conjecture, what the results may be. An example of a real lattice has been investigated by Pfander and Rashkov [40].

2. Since a symplectic structure is implicit in the commutation relations defining the adjoint lattice, one may wonder where symplectic analysis comes in. How do symplectic structure and complex structure interact?

3. What about non-uniform sampling sets $\Lambda \subseteq \mathbb{C}^d$ for $\mathcal{F}$? We are aware only of a qualitative results that follows from a more general theory: If the maximum distance to the nearest neighbor is small enough, then Λ is a set of sampling for $\mathcal{F}$. Precisely, there exists a constant $\delta > 0$ such that every relatively separated set Λ satisfying $\bigcup_{\lambda \in \Lambda} B(\lambda, \delta) = \mathbb{C}^d$ is a set of sampling for $\mathcal{F}$ [13].

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